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Comparison Between Dedicated Model Updating Methods and Hybrid Method

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By investigation of the short comings of two model updating methods, a hybrid method is suggested that make use of the advantages of each method and eliminates the disadvantage that eigenbased methods suffer from, an incomplete modal model. The main advantages of these methods are the small amount of data required and experimental noise elimination (minimization). On the other hand, using frequency reduced function (FRF) methods, one is not required to perform any modal analysis, and the data can be used directly, but theses methods suffer from the insensitivity of eigenvalues to FRF variations.

I. Introduction

BECAUSE of the increasing demands for better performance and the use of lighter structures in modern machinery, vibration engineers must have better testing and analysis tools than before. Particularly in the aerospace industries because of the importance and accuracy, the best methods for the correlation of analysis and tests should be found.

One of the most interesting subjects in structural dynamics is model updating. At the design stage, an analytical model, which is often a finite element one, can be used to demonstrate vibration behavior of the structure. At the next stage, after producing the structure, modal testing is used to validate the finite element model. If the model accurately demonstrates the dynamic behavior of the structure, it could be used for further analysis. However, test results are seldom in full agreement with the prediction of the finite element model. Therefore, the analyst and the experimentalist are faced with the problem of reconciling two modal databases for the same structure. Neither of these can be used alone to give an accurate description of the dynamics of the structure.

The best advantage of the finite element model, which is due to its large number of degrees of freedom, is its coverage of a wide frequency range. However because of geometrical simplification or element modeling, particularly at joints, it has considerable errors. On the other hand, experimental data or modal properties of the structure are real or at least close to real, because a real structure is tested and it is not an idealistic analysis.

In model updating, we try to acquire modal test results. Then in the finite element model, we can find the invisible degrees of freedom (DOF) such as internal DOF, rotational or other DOF that are too expensive to measure. Model updating methods could be broadly divided to two categories, namely, eigenbased methods and frequency reduced function- (FRF-) based methods. Each of these categories has its own advantages and disadvantages, which will be discussed briefly in the following section, and then the need for the development of a hybrid method will demonstrated.

II. Brief Introduction to Updating Methods

The two updating methods are compared and the governing equations of the FRF method are extracted. Then a new method is suggested to take advantage of both methods.

A. Eigenbased Methods

Eigenbased updating methods seek to update a given mass matrix $[M_A]$ and/or stiffness matrix $[K_A]$ using measured eigenvalues $[\Lambda_X]$ and eigenvectors $[\boldsymbol{\Phi}_X]$ under the equality constraints, such as eigendynamic and orthogonality properties. These methods can themselves be categorized into two groups by the types of variables to be updated. The first group uses individual elements of the system matrices as variables. The other group uses correction coefficients of the element matrices as variables.

1. *Berman's Method*[1]

Berman developed a method for updating both the analytical mass and the stiffness matrices.[1,2] The basic idea of the direct method is to minimize the weighted Euclidian norm between the original incorrect matrices and the updated ones under the equality considerations. In the Berman method, an analytical mass matrix is updated first, and then based on this updated mass matrix, the analytical stiffness matrix is updated. The objective functions to be minimized for updating mass and stiffnes matrices are

$$[M_U] = [M_A] + [M_A][\boldsymbol{\Phi}_X][m_a]^{-1}([I] - [m_a])[m_a]^{-1}[\boldsymbol{\Phi}_X]^T[M_A] \tag{1}$$

where

$$[m_a] = [\boldsymbol{\Phi}_X]^T[M_A][\boldsymbol{\Phi}_X] \tag{2}$$

The updated stiffness matrix becomes

$$[K_U] = [K_A] + ([\Delta] + [\Delta]^T) \tag{3}$$

where

$$[\Delta] = \tfrac{1}{2}[M_U][\boldsymbol{\Phi}_X]\big([\boldsymbol{\Phi}_X]^T[K_A][\boldsymbol{\Phi}_X] + \big[\omega_X^2\big]\big)[\boldsymbol{\Phi}_X]^T[M_U] - [K_A][\boldsymbol{\Phi}_X][\boldsymbol{\Phi}_X]^T[M_U] \tag{4}$$

This method does not require itration or eigenanalysis, and the updated model possesses the correct eigenvectors and eigenvalues. However, the updated mass matrix using Berman's method cannot preserve the connectivity of the structure because a connectivity constraint is not imposed. The updated stiffness matrix using the updated mass matrix, which is not correct, also cannot be correct. As a result, the updated model is not physically meaningful but is a representative model.

2. *Eigendynamic Constraint Method*

When correction coefficients of submatrices of system matrices are used as variables instead of individual elements of system matrices, the connectivity constraint can be easily imposed. Ibrahim developed a method that used submatrices of system matrices as variables under the eigendynamic constraint.[3] However, the updated model is not unique in the sense that it can be selected by an arbitrary

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factor. The method is formulated based on the eigendynamic equation and the mass normalization relationship. For the rth mode,

$$-\lambda_{xr}[M_U]\{\Phi_x\}_r + [K_U]\{\Phi_x\}_r = \{0\}$$
$$\{\Phi_x\}_r^T[M_U]\{\Phi_x\}_r = 1 \tag{5}$$

The updated mass and stiffness matrices can be written as

$$[M_u] = \sum_{j=1}^{L_2} a_j[M_j], \qquad [K_u] = \sum_{j=1}^{L2} b_j[K_j] \tag{6}$$

where a_j and b_j are correction factors to be determined and $[M]_j$ and $[K]_j$ are submatrices of the system matrices, such as 1) subelement matrices, 2) finite element matrices, and 3) macroelemnt matrices. When these equations are substituted in a set of N linear algebraic equations,

$$[-\lambda_{xr}[M]_1\{\phi_x\}_r \ldots - \lambda_{xr}[M]_{L1}\{\phi_x\}_r \quad [K_1]\{\phi_x\}_r \ldots [K]_{L2}\{\phi_x\}] \begin{Bmatrix} a_1 \\ \vdots \\ a_{L1} \\ b_1 \\ \vdots \\ b_{L2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{7}$$

Similarly,

$$\left[\{\phi_x\}_r^T[M]_1\{\phi_x\}_r \ldots \{\phi_x\}_r^T[M]_{L1}\{\phi_x\}_r \quad 0\ldots0\right] \begin{Bmatrix} a_1 \\ \vdots \\ a_{L1} \\ b_1 \\ \vdots \\ b_{L2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix} \tag{8}$$

When these equations are combined,

$$\begin{bmatrix} -\lambda_{xr}[M]_1\{\varphi_x\}_r & \ldots & -\lambda[M]_{L1}\{\varphi_x\}_r & [K]_1\{\varphi_x\}_r & \ldots & [K]_{L2}\{\varphi_x\}_r \\ \{\varphi_x\}_r^T[M]_1\{\varphi_x\}_r & \ldots & \{\varphi_x\}_r^T[M]_{L1}\{\varphi_x\}_r & 0 & \ldots & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_{L1} \\ b_1 \\ \vdots \\ b_{L2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{9}$$

When there are m modes available, we can have $m(N+1)$ linear algebraic equations,

$$\begin{bmatrix} [A_1] \\ [A_2] \\ \vdots \\ [A_m] \end{bmatrix} \{P\} = \begin{Bmatrix} \{C_1\} \\ \{C_2\} \\ \vdots \\ \{C_m\} \end{Bmatrix} \tag{10}$$

This method does not require itration or eigenanalysis, and the updated model preserves the connectivity of the structure. However, like other direct methods, mode expansion is essential because of the large difference in the dimensions between the measured modes and the analytical model.[4]

3. Inverse Eigensensitivity Method

One of the most applied and user-friendly methods to update the finite element model is the inverse eigensensitivity method (IEM), which approaches the experimental data by iteration and presents acceptable results. This method assumes that the mass and stiffness matrices of the system result from the summation of weighted element matrices,[5]

$$[M_U] = \sum_{i=1}^{L} a_i[M_i], \qquad [K_U] = \sum_{i=1}^{L} b_i[K_i] \tag{11}$$

This method is based on the Taylor expansion series of f function around $\boldsymbol{P}_0$:

$$f_i(\boldsymbol{p}) = f_i(\boldsymbol{p}_0) + \sum_{i=1}^{L} \frac{\partial f_i}{\partial p_i}\delta p_j + \sum_{j=1}^{L}\sum_{k=1}^{L} \frac{\partial^2 f_i}{\partial \boldsymbol{p}_j \partial \boldsymbol{p}_k} \Delta \boldsymbol{p}_i \Delta \boldsymbol{p}_k + \cdots \tag{12}$$

When the second and higher orders are neglected,

$$f_i(\boldsymbol{p}) - f_i(\boldsymbol{p}_0) = \sum_{j=1}^{L} \frac{\partial f_i}{\partial f_j} \Delta \boldsymbol{p}_j \tag{13}$$

or in matrix form

$$\begin{Bmatrix} f_1(p) - f_1(p_0) \\ \vdots \\ f_m(p) - f_m(p_0) \end{Bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \boldsymbol{p}_1} & \frac{\partial f_1}{\partial \boldsymbol{p}_2} & \cdots & \frac{\partial f_1}{\partial \boldsymbol{p}_L} \\ \frac{\partial f_m}{\partial \boldsymbol{p}_1} & \frac{\partial f_m}{\partial \boldsymbol{p}_2} & \cdots & \frac{\partial f_m}{\partial \boldsymbol{p}_L} \end{bmatrix} \begin{Bmatrix} \Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_L \end{Bmatrix} \tag{14}$$

The unknown vector $\boldsymbol{p}$ could be found from

$$\begin{Bmatrix} \Delta\lambda_1 \\ \{\Delta\varphi_1\} \\ \vdots \\ \Delta\lambda_m \\ \{\Delta\varphi_m\} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \lambda_1}{\partial a_1} & \ldots & \frac{\partial \lambda_1}{\partial a_L} & \frac{\partial \lambda_1}{\partial b_1} & \ldots & \frac{\partial \lambda_1}{\partial b_L} \\ \frac{\partial \{\varphi_A\}_1}{\partial a_1} & \ldots & \frac{\partial \{\varphi_A\}_1}{\partial a_L} & \frac{\partial \{\varphi_A\}_1}{\partial b_1} & \ldots & \frac{\partial \{\varphi_A\}_1}{\partial b_L} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \lambda_m}{\partial a_1} & \ldots & \frac{\partial \lambda_m}{\partial a_L} & \frac{\partial \lambda_m}{\partial b_1} & \ldots & \frac{\partial \lambda}{\partial b_L} \\ \frac{\partial \{\varphi_A\}_m}{\partial a_1} & \ldots & \frac{\partial \{\varphi_A\}_m}{\partial a_L} & \frac{\partial \{\varphi_A\}_m}{\partial b_1} & \ldots & \frac{\partial \{\varphi_A\}_1}{\partial b_L} \end{bmatrix} \begin{Bmatrix} \Delta a_1 \\ \vdots \\ \Delta a_L \\ \Delta b_1 \\ \vdots \\ \Delta b_L \end{Bmatrix} \tag{15}$$

or

$$\Delta = [S]\{\Delta p\} \tag{16}$$

where L is the number of elements. Other notes and constraints for these equations may be found in Ref. 5. The sensitivity values for λ_I and ϕ_i can be obtained from

$$\frac{\partial\lambda_r}{\partial a_i} = -\lambda_r\{\varphi\}_r^T[M]_i\{\varphi\}_r, \qquad \frac{\partial\lambda_r}{\partial b_i} = \{\varphi\}_r^T[K]_i\{\varphi\}_r$$

$$\frac{\partial\{\varphi\}_r}{\partial a_i} = \sum_{j=1}^{N}\alpha_{rj}^i\{\varphi\}_j, \qquad \frac{\partial\{\varphi\}_r}{\partial b_i} = \sum_{j=1}^{N}\beta_{rj}^i\{\varphi\}_j$$

$$\alpha_{rj}^i = \begin{cases} \dfrac{-\lambda_r\{\varphi\}_r^T[M]_i\{\varphi\}_r}{\lambda_r - \lambda_j}, & r \neq j \\ -\frac{1}{2}\{\varphi\}_j^T[M]_i\{\varphi\}_r, & r = j \end{cases}$$

$$\beta_{rj}^i = \begin{cases} \dfrac{\{\varphi\}_j^T[K]_i\{\varphi\}_r}{\lambda_r - \lambda_j}, & r \neq j \\ 0, & r = j \end{cases} \tag{17}$$

As is evident from Eq. (17) to have an exact $\partial\varphi_i/\partial\alpha_i$, one requires all of the modal vectors of the structure in all DOF, that is, modal and DOF completeness, which is usually not feasible for realistic structure.

Figure 1 shows a structure modeled by beam elements. Vertical and horizontal bars have five elements and the oblique bars modeled by seven elements. All of the elements have $E = 70E9$ Pa, $\nu = 0.3$, $\rho = 2700$ Kg/m^3, $A = 9e{-}4$ m^2, and $I = 6.75e{-}8$ m^4. Figure 2 shows the updated model using IEM. This method does not require mode expansion and because the connectivity constraint is imposed, it includes connectivity in updating procedure. It may be seen that, in this case, the updated model has very good accuracy with experimental data. The pitfalls of this method are related to calculating the sensitivity of the eigenvectors, which requires all of the modes.[6]

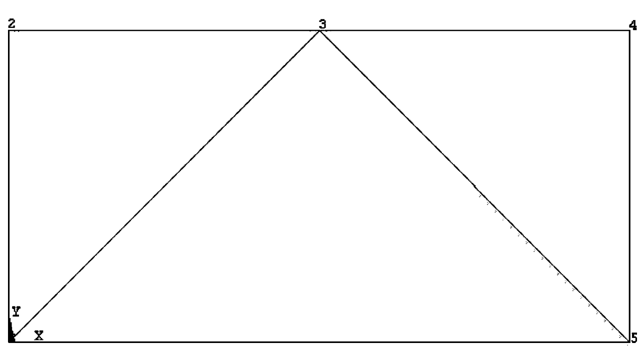

Fig. 1 Example of a beam element structure.

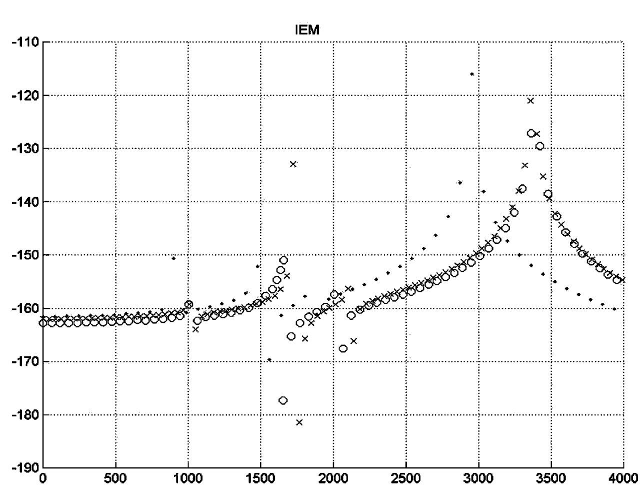

Fig. 2 IEM model updating: ●, analytical model; ×, experimental model; and ○, updated model.

B. FRF-Based Method

Natke presented the idea of FRFs first in 1988. Lin (Ref. 7) presented FRF method in his research (1990) based on output and input residuals[8]:

$$[Z_x(\omega, \{\varphi\})]\{X_x(\omega)\} = \{F_x(\omega)\}$$

$$[Z_A(\omega, \{\varphi\})]\{X_A(\omega)\} = \{F_A(\omega)\} \tag{18}$$

where $Z(\omega\phi)$ is dynamic stiffness matrix of the structure.

This method is based on the following mathematical equation[9]:

$$([A] + [B])^{-1} = [A]^{-1} - ([A] + [B])^{-1}[B][A]^{-1} \tag{19}$$

Assuming $[A]$ is $[Z_A(\omega)]$ and $([A] + [B])$ is $[Z_X(\omega)]$ and rearranging Eq. (18) in the form of receptance, we will have[8]

$$\{\alpha_A(\omega)\}_i^T - \{\alpha_X(\omega)\}_i^T = \{\alpha_X(\omega)\}_i^T[\Delta Z(\omega)][\alpha_A(\omega)] \tag{20}$$

If the last equation is written as Eq. (16)

$$\{\Delta\} = [C]\{p\} \tag{21}$$

then the elements of $[C]$ matrix could be found from

$$C_{sp} = -\omega^2\{\alpha_X\}_{s1}\left[\sum_{k=1}^{\mathrm{DOF}}\{\alpha_A\}_{sk}\sum_{r=1}^{\mathrm{DOF}}[M_p]_{kr}\right]$$

$$C_{s(p+1)} = \{\alpha_X\}_{s1}\left[\sum_{k=1}^{\mathrm{DOF}}\{\alpha_A\}_{sk}\sum_{r=1}^{\mathrm{DOF}}[K_p]_{kr}\right] \tag{22}$$

where $s,\ p = 1, \ldots, L$.

To demonstrate the insensitivity of eigenvalues to FRF variations, the results of a case study based on Eq. (21) is shown in Fig. 3. As is evident from Fig. 3, the FRF-based method has successfully shifted the FRF curves vertically to better agreement with experimental data, but horizontally it has failed to obtain experimental and analytical eigenvalues closer, which, in turn, indicate the earlier mentioned insensitivity.

As with the IEM, the Δp vector is calculated from singular value decomposition method, and the updated matrices are determined from Eq. (11) (Ref. 8).

Because FRFs are not sensitive to the eigenvalue variations and one can not find any sensible variation in the eigenvalues due to difference between experimental and analytical FRFs, there should be a suitable method to calculate updated matrices based on a combination of FRF and IE methods.

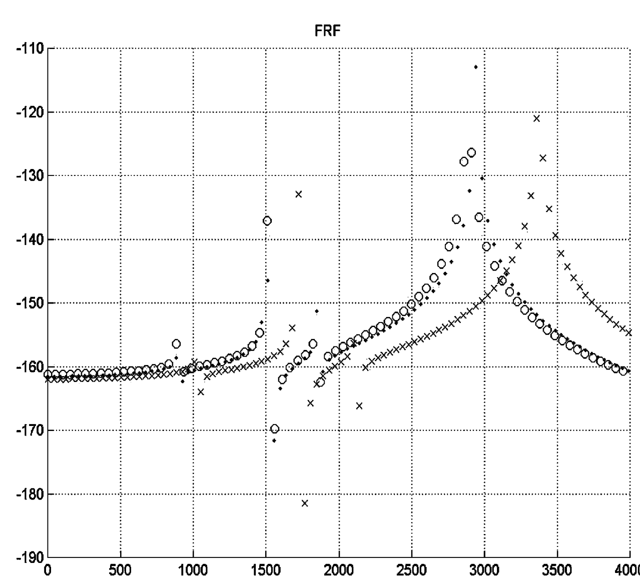

Fig. 3 FRF model updating: ●, analytical model; ×, experimental model; and ○, updated model.

III. Hybrid Method

Taking the advantage of eigenvalue sensitivity, which does not require any contribution for high modes, and adding it to the FRF method can eliminate the insensitivity disadvantage of the FRF method mentioned in Sec. II. To compensate for the insensitivity problem of the eigenvalues to FRF variations, Eq. (21) can be augmented by adding terms that directly relate the eigenvalues variations to structural parameters variation, that is, Eq. (17). This can be achieved by adding k natural frequencies sensitivity terms from the rows of Eq. (15) to the end of Eq. (21),

$$\begin{Bmatrix} \{B\} \\ -- \\ \dfrac{\Delta\lambda}{\lambda_1} \\ \vdots \\ \dfrac{\Delta\lambda_k}{\lambda_k} \end{Bmatrix} = \begin{bmatrix} & & [C] & & & \\ -- & -- & -- & -- & -- & -- \\ \dfrac{\partial\lambda_{A1}}{\partial a_1}\Big/\lambda_1 & \cdots & \dfrac{\partial\lambda_{A1}}{\partial a_L}\Big/\lambda_1 & \dfrac{\partial\lambda_{A1}}{\partial b_1}\Big/\lambda_1 & \cdots & \dfrac{\partial\lambda_{A1}}{\partial b_L}\Big/\lambda_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial\lambda_{Ak}}{\partial a_1}\Big/\lambda_k & \cdots & \dfrac{\partial\lambda_{Ak}}{\partial a_L}\Big/\lambda_k & \dfrac{\partial\lambda_{Ak}}{\partial b_1}\Big/\lambda_k & \cdots & \dfrac{\partial\lambda_{Ak}}{\partial b_L}\Big/\lambda_k \end{bmatrix} \times \{\Delta P\} \tag{23}$$

Equation (23) could be summarized as

$$\{\Delta_H\} = [S_H]\{\Delta P\} \tag{24}$$

where subscript H indicates the hybrid method.

Similar to IE and FRF methods, the hybrid sensitivity matrix ($[S_H]$) should be balanced.

Note that amongst other advantages of hybrid method, here one does not require any mode paring process as is required in IEM.

Figure 4 shows the results of hybrid method using Eq. (24). When results in Fig. 4 are compared with those in Fig. 3, it can be seen that most of frequencies the model are successful updated to experimental data. In this case, only five natural frequencies sensitivity terms are added to Eq. (23).

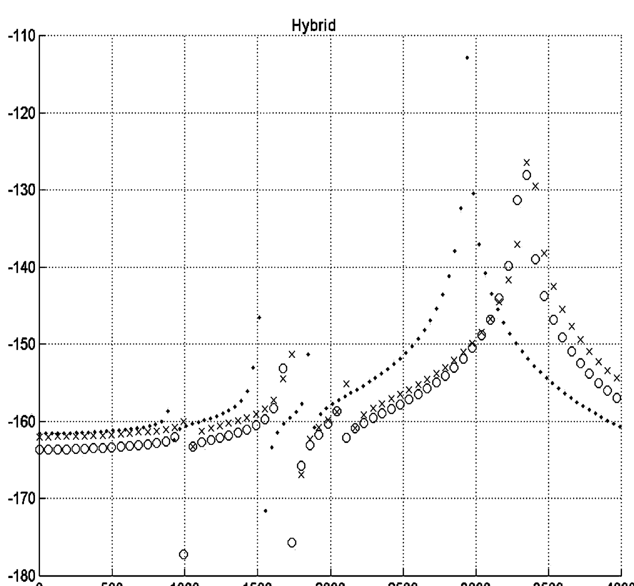

Fig. 4 Hybrid model updating: ●, analytical model; ×, experimental model; and ○, updated model.

A. Weighted Hybrid Method

The effectiveness of the method around various natural frequencies can be affected and adjusted considerably by applying proper weight to the related equation. This is demonstrated as

$$\begin{Bmatrix} \{\Delta_{\mathrm{FRF}}\} \\ --- \\ w_1\left(\dfrac{\Delta\lambda_1}{\lambda_1}\right) \\ w_2\left(\dfrac{\Delta\lambda_2}{\lambda_2}\right) \\ \vdots \end{Bmatrix} = [S_H]\{\Delta P\} \tag{25}$$

where w_n is a weighing coefficient.

Figure 5 shows the model with its third eigenvalue being weighted. When the results in Fig. 5 are compared with those in Fig. 4, it is evident that in this case there is more accuracy at third eigenvalue and the difference between updated model and experimental data is minimized. Because the third eigenvalue has 1.2 weighting coefficient, the fourth eigenvalue is oriented to the third one. Using this procedure, one can fine tune the updated model. Figures 6 and 7 show fine tuning of the FRF using several weighting coefficients.

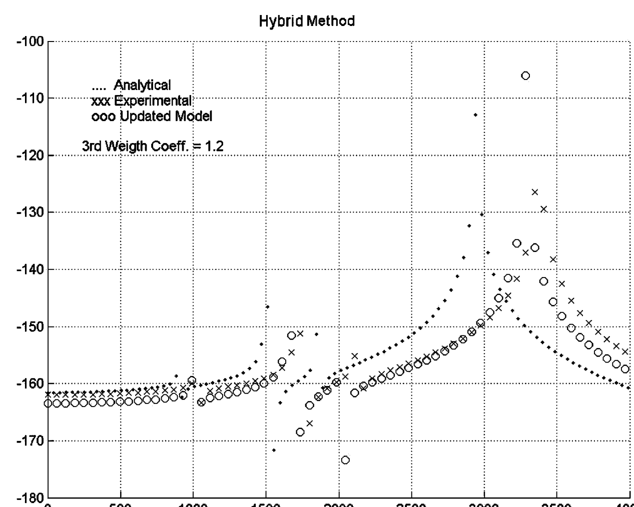

Fig. 5 WHM.

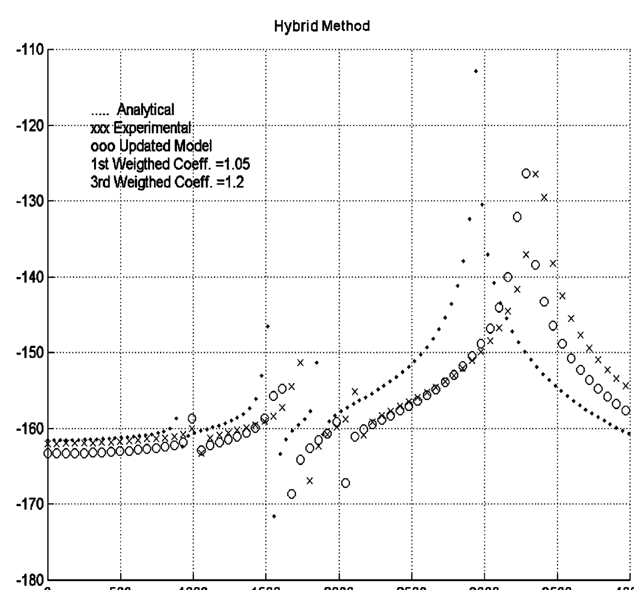

Fig. 6 Hybrid method fine tuning (2 weighted coefficients).

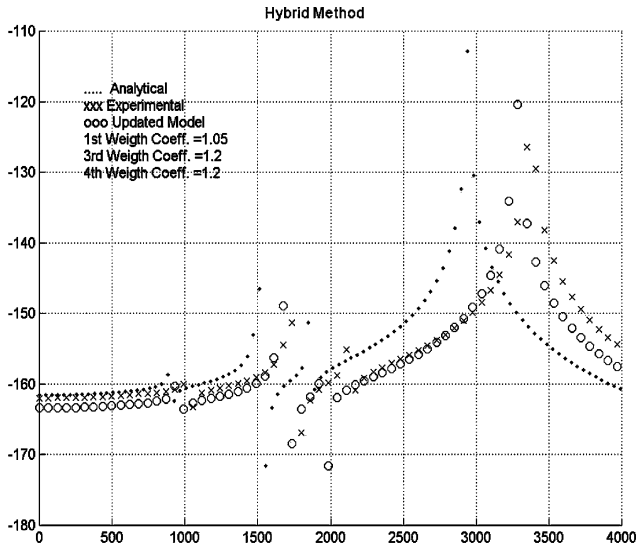

Fig. 7 Hybrid method fine tuning (3 weighted coefficients).

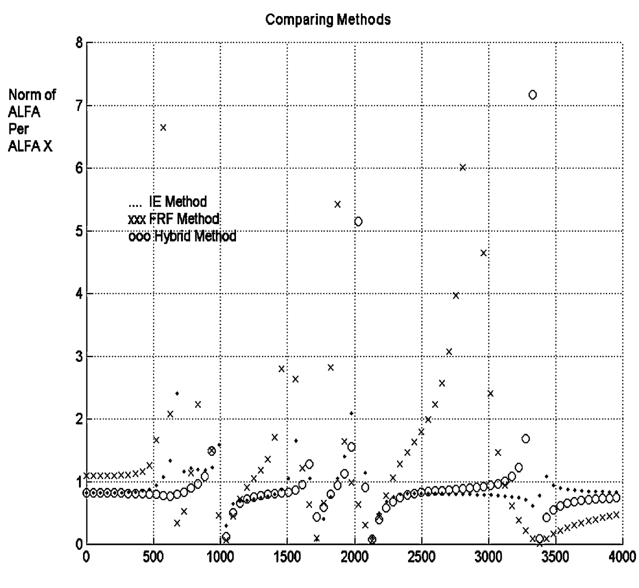

Fig. 8 Comparison between IE, FRF, and hybrid methods.

For a quantitative measure for the quality of the updating process, the following index is defined as the updating efficiency index (UEI):

$$\text{UEI} = \|\{H_{A1}\}| - |\{H_{x1}\}\|/|\{H_{x1}\}| \tag{26}$$

The UEI for each of IE, FRF, and hybrid methods are shown in Fig. 8. Note that at all frequencies the value of the UEI for the hybrid method is very close to the IEM.

Note that weighted functions also give an indication of the structural dynamic behavior sensitivity to a particular eigenvalue variation. The hybrid method is very sensitive to the variation of the coefficients, and, for example, a 0.001 change in coefficients can affect the FRF plot considerably.

B. Noise Effects on IE, FRF and Hybrid Methods

All model updating methods suffer from noise effects, which can never be eliminated completely. These noises originate from several sources: 1) experimental errors, such as shaker/structure interactions, experimentalist errors, instruments, signal processing, leakage, etc., and 2) computational errors, such as circle fit or line fit errors, numerical errors, round off errors, etc. In eigenbased methods because the eigenvalues and eigenvectors are extracted by curve-fitting methods, which in a way filter the effect of noise, they generally are not catastrophic, but in FRF methods because the raw

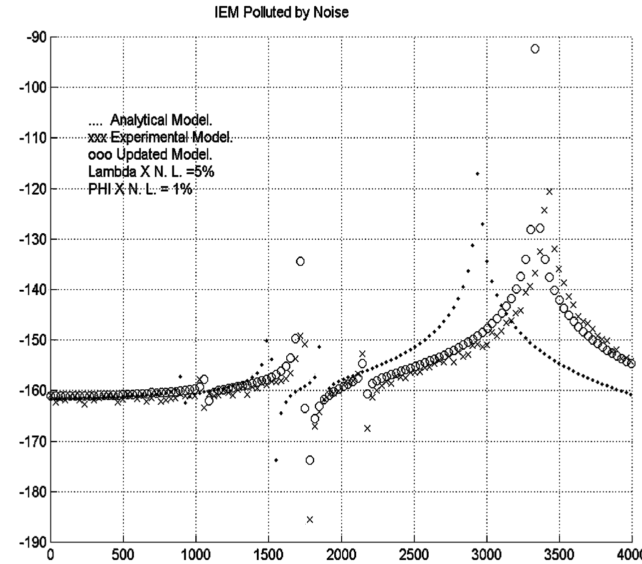

Fig. 9 Noise effects on IE method (first noise level).

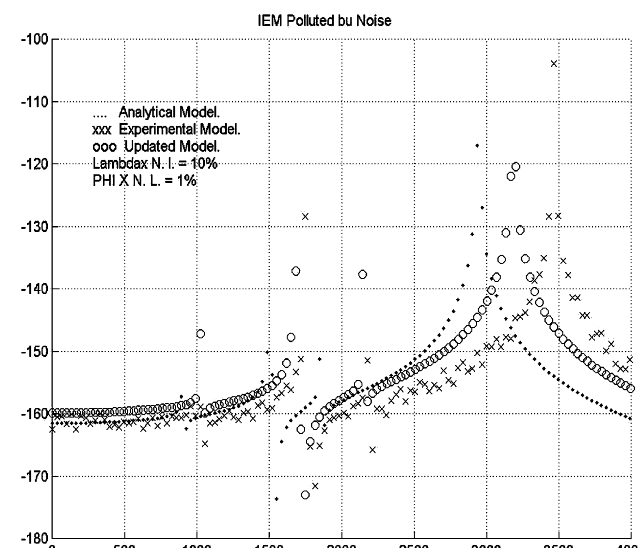

Fig. 10 Noise effects on IE method (second noise level).

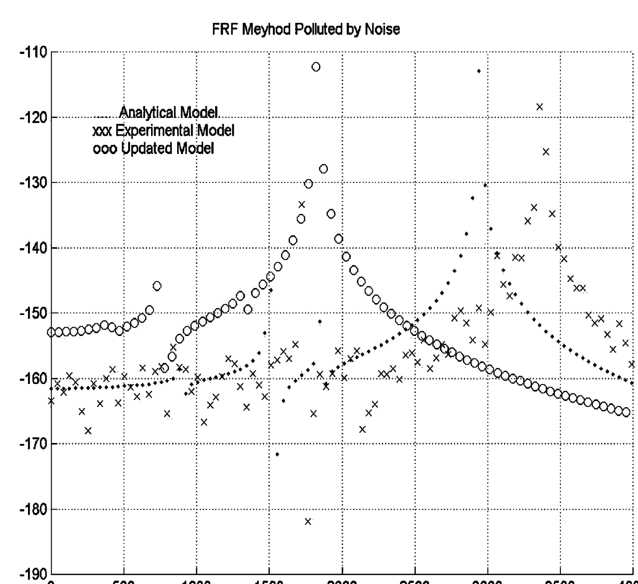

Fig. 11 Noise effects on FRF method (first noise level).

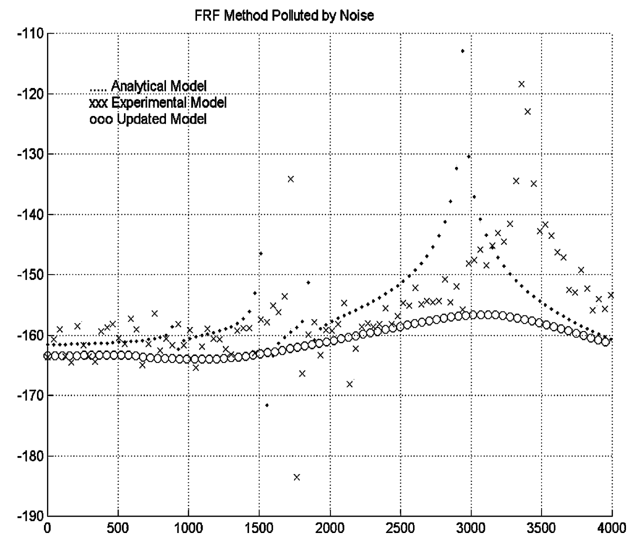

Fig. 12 Noise effects on FRF method (second noise level).

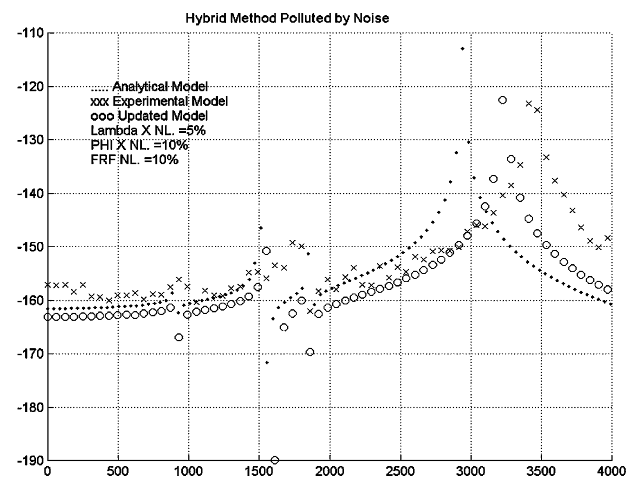

Fig. 13 Noise effects on hybrid method (first noise level).

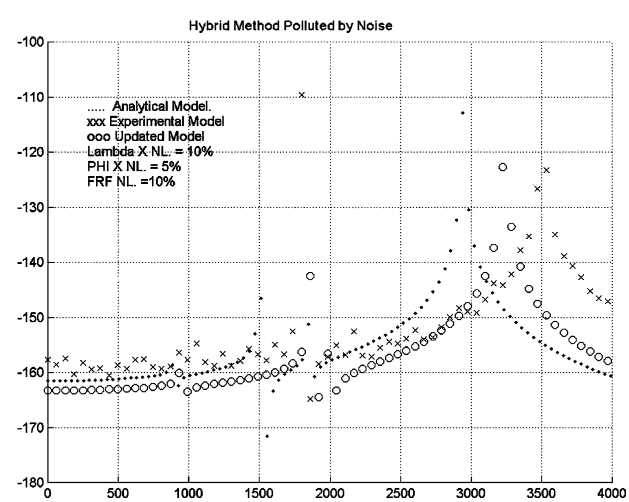

Fig. 14 Noise effects on hybrid method (second noise level).

FRFs are being used, the effect of noise could be considerable. Figures 9 and 10 show the noise effect with the IEM, but with the FRF method, the iteration may never converge for polluted data (Figs. 11–13).

The hybrid method is least affected by the noise (Fig. 14) because the eigenvalues are the least affected by noise, modal parameters (comparing to eigenvectors and modal damping). This is shown in Fig. 15.

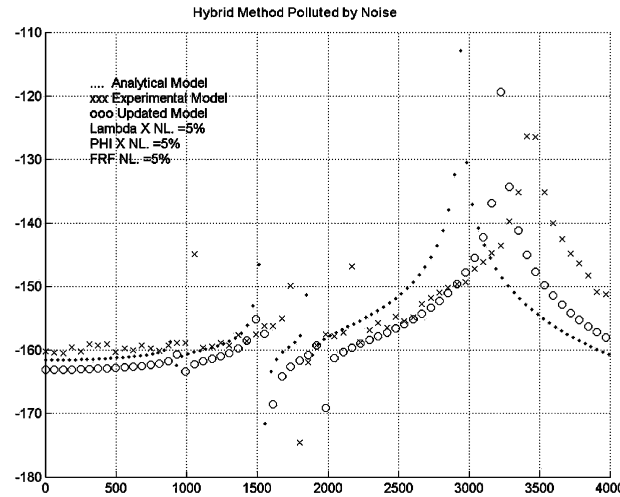

Fig. 15 Noise effects on hybrid method (third noise level).

IV. Conclusions

The weighted hybrid method (WHM) as formulated here can be considered as a method that provides the analyst great flexibility in using the advantages of the FRF and IE methods and minimizing their disadvantages, especially regarding the noise effects. In contrast to the FRF method, the WHM gives the analyst the power of zooming in the frequency domain and, thus, performing localized fine tuning around natural frequencies.

One of the advantages of model updating using eigensensitivity analysis is that mode expansion (or reduction) is not required. However, this method requires a large computational effort because of repeated solution of the eigendynamic problem and repeated calculation of the sensitivity matrix.

Because the IEM is a multivariable Newton–Ralphson method, the convergence of the IEM can be improved by introducing error location procedure and by setting bounds on $\{\Delta p\}$ (Ref. 8).

Both the IE and FRF methods discussed in this paper satisfy the connectivity considerations. Therefore, the hybrid method satisfies them as well.

The noise effect on the IE, FRF and hybrid methods is studied. Although the hybrid method is based on the FRF, the noise effects in the hybrid method are negligible, and they could be reduced by fine tuning the FRF plot.

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